MMP Learning Seminar.

Week 51.

Content:

Semi-log canonical abundance.

Finiteness of B-representations:

Theorem 1.2: projective dlt pair (X,Δ) s.t $K_X+\Delta$ is semiample and Δ is a $\mathbb{Q}$-divisor. There exists $m \in \mathbb{N}$ such that the image of the B-representation

$$\rho_M : \mathrm{Bir}(X,\Delta) \longrightarrow \mathrm{Aut}(H^0(X,\mathcal{O}_X(M(K_X+\Delta))))$$

is finite for M divisible by m

Abundance for slc pairs:

Theorem 1.4: Let (X, Δ) be a slc pair.

$f: X \longrightarrow S$ projective morphism. $n: \bar{X} \longrightarrow X$ normalization

$n^*(K_X + \Delta) = K_{\bar{X}} + \bar{\Delta} + \bar{D}$, where $\bar{D}$ is the double locus.

If $K_{\bar{X}} + \bar{\Delta} + \bar{D}$ is semiample over S, then $K_X + D$ is semiample over S.

Corollary 1.5: Let (X, Δ) be a $\mathbb{Q}$-factorial dlt pair which is projective over a base S, and $T := \lfloor \Delta \rfloor$, where Δ is a $\mathbb{Q}$-divisor. Suppose that:

(1) $K_X + \Delta$ is nef over S,

(2) $(K_X + \Delta)|_{T_i}$ is semiample over S for each component $T_i \subseteq T$.

(3) $K_X + \Delta - \varepsilon P$ is semiample over S for some $\mathbb{Q}$-div $P \geq 0$ with $\mathrm{supp}(P) = \mathrm{supp}\, T$ and $\varepsilon > 0$ sufficiently small.

then $K_X + \Delta$ is semiample over S.

Corollary 1.6: (X, Δ) is slc $f: X \longrightarrow S$ a projective morphism s.t $K_X + \Delta \equiv_{\mathbb{Q}, S} 0$, then $K_X + \Delta \sim_{\mathbb{Q}, S} 0$.

$\boxed{\text{Aim: } K_X + \Delta \text{ semiample over } S.}$

Log canonical stratifications:

$f:(X,\Delta)\longrightarrow Z$ crepant log structure

$K_X+\Delta\sim_{f,\mathbb{Q}}0.$

$S_i^*(Z,X,\Delta)\subseteq Z$ the union of all $\leq i$-dimensional lc centers of f.

Set $S_i(Z):=S_i^*(Z,X,\Delta)\setminus S_{i-1}^*(Z,X,\Delta)$

$S_i(Z)$ are locally closed subspaces of Z of pure dim i.

$Z=\bigsqcup_i S_i(Z)$, $S_{\dim(Z)}(Z)$ the open stratum.

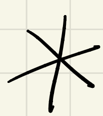
The union of the $S_i(Z)$ is called the lc stratification of Z.

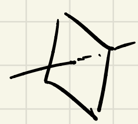
(Z,S^*) its boundary is $B(Z,S_*):=Z\setminus S_{\dim(Z)}(Z)$.

An lc center of (Z,S^*) is the closure of an irreducible component of some stratum of $S_i(Z)$.

Lemma: $S_i(Z)$ are unibrach and $B(Z,S_*)$ is seminormal.

$\hat{T}_y$ irreducible


$\subseteq \mathbb{A}^n$


$\mathbb{A}^n$

unibranch + seminormal $\Leftrightarrow$ normal

Lemma: $f : (X, \Delta) \to Z$ dlt log crepant structure and (Z, S^*) the lc strata. $Y \subseteq X$ lc center.

$Y \xrightarrow{f_Y} W \longrightarrow Z$ Stein factorization

$f_Y : \underset{(Y, \Delta_Y)}{Y} \longrightarrow W$ is a crepant log structure.

$$S_i(W) = \pi^{-1}(S_i(Z)) \text{ for all } i.$$

Definition: (Z_j, S_*^j) be two stratified spaces. A finite morphism $\pi : Z_1 \longrightarrow Z_2$ is said to be stratified if the following equivalent conditions hold:

(1) $S_i^1(Z_1) = \pi^{-1}(S_i^2(Z_2))$ for all i,

(2) for every component $W_{ij} \subseteq S_i^j(Z_j)$, its image $\pi(W_{ij})$ is a component of $S_i^2(Z_2)$.

Definition (stratification of lc origin).

(Y, S^*) stratified space is of lc origin if the following conditions hold:

(1) all the strata $S_i(Y)$ are unibranch.

(2) there are crepant structures $f_j : (X_j, \Delta_j) \longrightarrow Z_j$ with lc stratifications (Z_j, S_*^j) and a finite, surjective, stratified morphism $\pi: \coprod_j (Z_j, S_*^j) \longrightarrow (Y, S_*)$.

Remark: $f: (X, \Delta) \longrightarrow W$ is a crepant structure, $Y \subseteq W$ union of lc centers, $S_i(Y) := Y \cap S_i(W)$ the seminormalization of (Y, S_*) is of lc origin

Definition (hereditary log canonical centers):

(Y, S_*) stratified space of lc origin.

The hereditary lc centers of (Y, S_*) are defined as follows:

(1) $W \subseteq Y$ a lc center $W^n \xrightarrow{\tau} Y$ normalization $S_i(W^n) = \tau^{-1}(S_i(Y))$. Then (W^n, S_*) is a hereditary lcc.

(2) If $W \neq Y$, then every hereditary lcc of (W^n, S_*) is also a Hlcc of (Y, S_*).

Remark: Each Hlcc (W, S^*) is a normal, stratified space of log canonical origin. and comes with a stratified finite morphism $\tau : (W, S^*) \longrightarrow (Y, S^*)$

Gluing relations: $\pi_j : (W_j, S^*) \longrightarrow (Y, S^*)$ finite set of hlcc. $\tau_{ijk} : W_i \longrightarrow W_j$ stratified isomorphisms.

$R(\tau) \rightrightarrows Y$ on geometric points of Y generated by the relations $\pi_i(w) \sim \pi_j(\tau_{ijk}(w))$ $\forall\, w \longrightarrow \coprod_i W_i$.

The geometric quotient is a seminormal stratification (X, S^*).

Theorem: (Y, S^*) stratified space $\boxed{\text{of lc origin}}$ and $R(T)$ a gluing relation on (Y, S^*). Assume that the $R(T)$-equivalence classes are all finite. Then the geometric quotient:

$$
\begin{array}{rccc}
 & & & X \\
 & & & \shortparallel \\
q : & (Y, S^*) & \longrightarrow & (Y/R(T),\, q_* S^*) \\
 & HN & \longrightarrow & HN \\
 & HSN & \longrightarrow & HSN.
\end{array}
$$

exists and $(Y/R(T), q_* S^*)$ is a stratified space $\boxed{\text{of lc origin}}$

log canonical stratifications:

N: (X, S^*) is (N) if each strata is normal.

SN: Both X and BX are both seminormal.

HN: X is (N) + $\pi: X^n \longrightarrow X$ stratifiable + $B(X^n)$ is (HN)

HSN: X is (SN) + $\pi: X^n \longrightarrow X$ stratifiable + $B(X^n)$ is (HSN)

Remark: A stratification of lc origin is HSN.

Corollary: $(\tilde{X}, \tilde{\Delta} + \tilde{D})$ lc, $\tilde{\tau}: \tilde{D}^n \longrightarrow \tilde{D}^n$ involution.

Assume $\tilde{\tau}$ maps lce of $(\tilde{D}^n, \mathrm{Diff}_{\tilde{D}^n}\tilde{\Delta})$ to lcc's.

and $R(\tilde{\tau}) \rightrightarrows \tilde{X}$ has finite equivalence classes.

S_2 + at most nodes in codimension 1

Then, there exists a demi normal pair (X, Δ) with

normalization $(\bar{X}, \bar{D} + \bar{\Delta}, \tau)$ such that.

(1) $(\tilde{X}, \tilde{\Delta} + \tilde{D}, \tilde{\tau}) = (\bar{X}, \bar{\Delta} + \bar{D}, \tau)$ and

(2) $R(\tilde{\tau}) = \mathrm{red}\,(\tilde{X} \times_X \tilde{X})$

Towards the proof of 1.4:

$$\begin{tikzcd} \bar X \arrow[rr, "\bar g"] \arrow[dr] & & \bar Y \arrow[dl] \\ & S & \end{tikzcd}$$

induced by $K_{\bar X} + \bar\Delta + \bar D$.

$(\sigma_1, \sigma_2) : \bar T \rightrightarrows \bar Y$.

$\bar T = \lfloor \bar\Delta \rfloor$

image of $D^n \rightrightarrows \bar X$.

$D^n :=$ normalization of $\bar D$.

$$\begin{tikzcd} D^n \arrow[r] \arrow[d] & \bar X \arrow[r] \arrow[d, "\varphi"] & X \arrow[d, "\varphi_Y"'] \\ \bar T \arrow[r, hook] \arrow[dr] & \bar Y \arrow[r] \arrow[d] & Y \arrow[dl] \\ & S & \end{tikzcd}$$

$K_{\bar X} + \bar\Delta + \bar D = \varphi^* A$

$K_X + \Delta = \varphi_Y^* A_Y$

$S, \bar T$ and $S, \bar Y$.

satisfies (HN) and (HSN).

Aim: $\bar T \rightrightarrows \bar Y$ generates a finite relation $R \rightrightarrows \bar Y$.

It suffices to check this finiteness over generic points of strata V° of $S, \bar Y$.

We must check $R \times_S \bar\eta \subseteq \bar Y \times \bar Y \times_S \bar\eta$ is finite over $V^\circ \times \eta$, $\eta =$ generic point of $f(V_0)$

Proposition: Minimal lcc's are birational.

$$\begin{tikzcd} Z \arrow[r] & \tilde{V} \arrow[r] & V \\ Z' \arrow[u, "\mathrm{bir}"] \arrow[r] & \tilde{V}' \arrow[u, phantom, "\wr"] \arrow[ur] & \end{tikzcd}$$

source spring

Theorem: Let $R \subseteq \bar{T} \times \bar{T}$ be the relation generated by $\bar{T} \rightrightarrows \bar{T}$. $p_i : \mathrm{Spr}_i(\bar{T}, X^d, \Delta^d) \longrightarrow S_i\bar{T}$ be the induced finite morphisms. $\bar{\eta}_{ij}$ = alg closure of the generic point of $f(V_{ij})$. Then

$$\left((p_i \times p_i)^{-1}\left(R \cap (S_i\bar{T} \times \overline{S_i\bar{T}})\right) \times_S \bar{\eta}_{ij}\right) \cap \left(\tilde{V}_{ij}^{\circ} \times \tilde{V}_{ij}^{\circ} \times_S \bar{\eta}_{ij}\right)$$

is contained in the graph $\bigcup_g \Gamma(\chi(g))$ for all

$$g \in \mathrm{Bir}\left(Z_{\bar{\eta}_{ij}}, \mathrm{Diff}^{*}_{Z_{\bar{\eta}_{ij}}} \Delta^d\right).$$

Proof of 1.4: By finiteness of B-representations, $\bigcup_g \Gamma(\chi(g))$ is finite, hence $R \rightrightarrows \bar{T}$ is finite $\square$.

Proof of Proposition 1.5:

T_i comp of $T = \lfloor \Delta \rfloor$.

$K_{T_i} + \Delta_{T_i} = K_X + \Delta|_{T_i}$ is semiample

$$T \leftarrow \bigsqcup_i T_i$$

By Thm 1.4 $K_T + \Delta_T$ is semiample $\lrcorner$

$Bs(m(K_X+\Delta))$ must be contained in the support of T.

$$m(K_X+\Delta) - T = \underbrace{(m-1)(K_X+\Delta-\varepsilon P)}_{\text{semiample}} + \underbrace{K_X+\Delta-(T-(m-1)\varepsilon P)}_{\text{klt}}$$

Kollár's injectivity: $R^1 f_* \mathcal{O}_X(m(K_X+\Delta)-T) \longrightarrow R^1 f_* \mathcal{O}_X(m(K_X+\Delta))$

$$\begin{tikzcd}
f^* f_* \mathcal{O}_X(m(K_X+\Delta)) \arrow[r, two heads] \arrow[d] & f^* f_* \mathcal{O}_T(m(K_T+\Delta_T)) \arrow[d, two heads] \\
\mathcal{O}_X(m(K_X+\Delta)) \arrow[r, two heads] & \mathcal{O}_T(m(K_T+\Delta_T))
\end{tikzcd}$$

semiample along T over S. semiample over S.

$\square$

Proof of 1.6:

Thm 1.4 $\longrightarrow$ Assume (X,Δ) is lc

$\xrightarrow{\text{dlt mod}}$ Assume (X,Δ) $\mathbb{Q}$-fact + dlt.

$$T = \lfloor \Delta \rfloor$$

(T,Δ_T) is sdlt.

$(K_X+\Delta-\varepsilon T)$ – MMP with scaling of an ample. over S.

If it ends with a MFS then we proceed inductively.

It ends with a minimal model $\Longleftrightarrow$ every comp of T is vertical

$K_X+\Delta \sim_{\mathbb{Q}} 0$ on the general fiber.

klt on the general fiber.

$(X,\Delta-\varepsilon T)$ has a gmm over the general fiber $\Longrightarrow$

$(X,\Delta-\varepsilon T)$ has a gmm $(X',\Delta'-\varepsilon T')$ over S.

(X',Δ') is lc, $\quad K_{X'}+\Delta' \equiv_S 0. \quad -\varepsilon T'$ semiample over S.

$K_{X^d}+\Delta^d \equiv_S 0$

$(X^d,\Delta^d) \longrightarrow (X',\Delta')$ dlt mod.

$$\begin{tikzcd}
 & (X^d,\Delta^d) \arrow[d] \\
X \arrow[r, dashed] \arrow[d] & X' \arrow[dl] \\
S &
\end{tikzcd}$$

$(K_X+\Delta)$-trivial.

$\boxed{P = T^d} = \mu^* T', \quad \lfloor \Delta^d \rfloor = \operatorname{supp} T^d.$

Σ is a component $\lfloor \Delta^d \rfloor$.

$K_\Sigma + \Delta_\Sigma \equiv_S 0 \xrightarrow{\text{ind on dim}} K_{\Sigma'}+\Delta_{\Sigma'}$ semiample

1.5 $\Longrightarrow K_{X^d}+\Delta^d$ semiample

$\Longrightarrow K_{X'}+\Delta'$ semiample

$\Longrightarrow K_X+\Delta$ semiample $\quad$ $\Big\}$ over S $\quad \square$